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REPORT No. 907

On The Choice Of Mesh In The Integration Of Ordinary Differential Equations

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DEPARTMENT OF THE ARMY PROJECT No. 503-06-002
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT No. TB3-0007K

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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BGarfinkel/ddh
Aberdeen Proving Ground, Md.
April 1954

ON THE CHOICE OF MESH IN THE INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS

ABSTRACT

The optimum mesh at point x is defined here as the interval, h(x), which minimizes the time of integration of an ordinary differential equation while maintaining a prescribed bound of the error. An attack upon the problem of constructing such a mesh is carried out here with the aid of the Calculus of Variations. The Euler equations are derived and discussed, and the results are extended to a system of differential equations.

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INTRODUCTION

In the numerical integration of ordinary differential equations by a step-by-step method it is sometimes advantageous to change the step, h, as the integration proceeds. This suggests the following problem: "Given the equation

$$y' = f(x,y),\; y(0) = y_0,\; 0<x<X \tag{1}$$

and a numerical method whose local truncation error is of order h^k, $k \geq 2$, required the function $h(x)$, such that 1) the accumulated error, δy, does not exceed a prescribed bound, E^*, and 2) the time of integration is minimized".

Provided δy is sufficiently small, it will satisfy the variational equation

$$\frac{d}{dx}\,\delta y = f_y \delta y + \eta/h, \tag{2}$$

where η is the local error per step. If ϵ is an upper bound of the rate of the local error; i.e.,

$$|\eta/h| \leq \epsilon, \tag{3}$$

then δy has an upper bound, $E(x)$, satisfying the differential equation[1)]

$$E' - f_y E - \epsilon = 0, \quad E(0) = 0, \tag{4}$$

whose solution is

$$E(x) = \int_0^x \exp \int_\xi^x f\;(\alpha, y(\alpha))\, d\alpha\, \epsilon(\xi)\, d\xi. \tag{5}$$

A bound ϵ_1, of the error rate can be estimated as

$$\epsilon_1 = R/h + h^{k-1}\psi_k + h^k \psi_{k+1} + \ldots, \tag{6}$$

Where R is a bound of the local rounding error and $\psi_k(x), \psi_{k+1}(x)$ are functions of the zeroth order in h. If

$$R \ll (k-1)\psi_k^{k+1}/\psi_{k+1}^{k} \tag{7}$$

it is possible to choose h such that

$$h_m \equiv \left| R/(k-1)\,\psi_k \right|^{1/k} < h \ll \psi_k/\psi_{k+1}. \tag{8}$$

Then the second term in (6) dominates the third, and $\epsilon_1 \leq kh^{k-1}\psi_k$, so that a new bound can be conveniently chosen as $\epsilon = kh^{k-1}$, the subscript k of ψ having been dropped. If the rounding error is negligible, a more practical bound would be

$$\epsilon = h^{k-1}\psi \quad , \tag{9}$$

which we shall adopt here. On the other hand, if the rounding error is not negligible, the results to be derived will remain valid provided ψ is replaced by $k\,\psi$ wherever it occurs.

The simple problem of minimizing $E(x)$, regardless of the time consumed, is seen from (5) to be equivalent to that of minimizing $\epsilon_1 = R/h + h^{k-1}\psi$. Two properties of the solution, furnished by the function $h = h_m(x)$, may be noted in passing: 1) $\epsilon_1(h_m) = kh_m^{k-1}\psi$, 2) $R : h_m^k\psi = (k-1) : 1$; i.e. the local error bound is partitioned between the rounding and the truncation in the ratio $(k - 1) : 1$. The latter result[2] should be contrasted with the popular notion that this ratio should not exceed unity.

The function $\psi(x)$ is listed below for some of the more commonly used methods of step-by-step integration.

Table 1

Method	k	x	Remarks
Euler	2	$\frac{1}{2}\ y'''$	
Modified Euler[3]	3	$\frac{1}{12}\ y''' \ - \frac{1}{4}\ f_y y'''$	one iteration
" "	3	$\frac{1}{12}\ y'''$	two iterations
Kutta	4		
Runge-Kutta[4]	5	$c(x)y^{(5)}$	

The modified Euler method with one iteration, called by some the Heun method, can be cited as an exception to the general rule that ψ is of the form

$$\psi(x) = c(x)y^{(k)}(x), \tag{10}$$

where $y^{(k)}$ is the derivative of order k and $c(x)$ depends on $f(x,y)$ and the method used.

If new variables $H(x)$, $S(x)$ are now defined as

$$H(x) = \left| \exp\left(-\int_0^x f_y(\xi)\,d\xi\right) \psi(x)/\psi(0) \right|^{\frac{1}{k}}, \tag{11}$$

$$S(x) = \exp\left(-\int_0^x f_y(\xi)\,d\xi\right) E(x), \tag{12}$$

(4) can be rewritten

$$S' - \psi(0) h^{k-1} H^k = 0, \quad S(0) = 0. \tag{13}$$

The time t, of integration for a given method is proportional to the number of integration steps. Therefore,

$$t(x) = \int_0^x d\xi / h(\xi), \tag{14}$$

or

$$t' - h^{-1} = 0. \tag{15}$$

the three unknown functions h, S, and t are thus connected by two non-holonomic constraints (13) and (15). The system therefore has one degree of freedom, which can be realized by an arbitrary choice of $h(x)$.

VARIATIONAL APPROACH

We identify our problem with the Problem of Bolza[5] in the Calculus of Variations: "Required the function $h(x)$ in the prescribed range $(x_1 = 0,\ x_2 = X)$, which minimizes the function $g = t(x_2)$ and satisfies the differential constraints

$$\begin{aligned} \phi_1 &= \psi(0) h^{k-1} H^k(x) - S' = 0, \\ \phi_2 &= h^{-1} - t' = 0, \end{aligned} \tag{16}$$

subject to the end-conditions

$$S(0) = t(0) = 0, \quad S(X) = S^* \equiv E^* \exp\left(-\int_0^X f_y(\xi)\,d\xi\right)." \tag{17}$$

The Lagrangian,

$$F = \lambda_1 \phi_1 + \lambda_2 \phi_2, \tag{18}$$

must satisfy the Euler equations

$$\frac{d}{dx} F_{z_i'} = F_{z_i} \; ; \; i = 1, 2, 3, \tag{19}$$

with

$$z_1 = h, \; z_2 = S, \; z_3 = t. \tag{20}$$

Then (19) becomes

$$\begin{aligned} &\lambda_1 (k-1) \psi(0) h^{k-2} H^k - \lambda_2 h^{-2} = 0, \\ &\lambda_1 = \text{const.}, \\ &\lambda_2 = \text{const.}, \end{aligned} \tag{21}$$

leading immediately to

$$h = h(0) H^{-1} \tag{22}$$

$$h(0) = \left| \lambda_2 / (k-1) \; \psi \; (0) \lambda_1 \right|^{\frac{1}{k}} = \text{const.}$$

The parameter $h(0)$ is determined from the end-conditions as

$$h(0) = \left| S^* / \psi \; (0) \int_0^X H \; (\xi) \; d\xi \right|^{\frac{1}{k-1}}, \tag{23}$$

and $t(X)$ from (22.1), (16.2) as

$$t(X) = \int_0^X H \; (\xi) \; d\xi \; / h(0). \tag{24}$$

It can be shown that $t < \bar{t}$, where $\bar{t}$ corresponds to a constant interval, $\bar{h}$. For we have from (16)

$$\bar{h} = \left| S^* / \psi \; (0) \int_0^X H^k \; (\xi) d\xi \right|^{\frac{1}{k-1}}, \tag{25}$$

$$\bar{t} = X / \bar{h}.$$

Hence, with the and of the Hölder inequality[6] for $k \geq 1$, there follows

$$t : \bar{t} = \left| \bar{H}^{k} \Big/ \overline{H^{k}} \right|^{\frac{1}{k-1}} < 1, \tag{26}$$

where $\bar{H}$ and $\overline{H^k}$ are the mean values of H and H^k, respectively, on the interval $(0,X)$. This result, of course, would be expected if our solution furnishes the absolute minimum of t.

From (16) and (22.1) can be deduced

$$dS/dt = \text{const.} \tag{27}$$

$$E(x) = t \exp \int_0^x fy\,(\xi)d\xi\,\text{const.},$$

which admits of the following interpretation: "All points of an optimum mesh make equal contributions to the final truncation error".

THE EFFECT OF DISCONTINUITIES

Since the use of a continuous h(x) is physically impossible, it may be replaced by some convenient step-function, such as

$$h_*(x) = h_*\,(0)2^m,\ m = 0, \pm 1, \pm 2, \ldots, \tag{28}$$

with the additional constraint

$$\phi_3 = m' = 0, \tag{29}$$

the augmented Lagrangian,

$$F = \lambda_1\,(\Psi(0)h^{k-1}H^k - S') + \lambda_2(h^{-1} - t') + \lambda_3 m' \quad , \tag{30}$$

must satisfy the Corner Condition[7],

$$\Delta F_{z_i'} = 0, \quad \Delta(F - z_i'\ F_{z_i'}) = 0, \tag{31}$$

where Δ denotes a "jump"; e.g.,

$$\Delta F = F_+ - F_-. \tag{32}$$

From (31) and (30) is obtained

$$\Delta\lambda_1 = \Delta\lambda_2 = \Delta\lambda_3 = 0$$

$$\Delta(\lambda_1\ \Psi(0)h^{k-1}\ H^k + \lambda_2 h^{-1}) = 0. \tag{33}$$

If f is of class C^{k-1}, then

$$\Delta f_y = 0, \Delta H = 0, \tag{34}$$

and we deduce, with the aid of (28),

$$\Delta 2^{m(k-1)} / \Delta 2^{-m} = \text{const. } H^{-k} \tag{35}$$

Finally, if $m(0) = 0$ and $|\Delta m| = 1$, (35) yields $H^k 2^{-m} = 1$ at corners, and

$$m(x) = - \left[\log H(x)/\log 2 \right] \tag{36}$$

holding everywhere, the symbol [ ] denoting the integral part of a number $h(0)$ and $t(X)$ can now be calculated from (17) as

$$h_*(0) = \left| S^*/\psi(0) \int_0^X 2^{(k-1)m(\xi)} H^{k}(\xi) d\xi \right|^{\frac{1}{k-1}}$$
$$t_* = \int_0^X 2^{-m(\xi)} d\xi / h_* (0). \tag{37}$$

THE EFFECT OF INEQUALITIES

Since R, ψ_k, ψ_{k+1} occurring in (8) are generally difficult to calculate, some other convenient bounds may be imposed on h in the general form

$$a \leq h(x) \leq b. \tag{38}$$

Furthermore, instead of prescribing the terminal error bound, $E(X)$, there may be imposed a bound on the function $E(x)$:

$$E(x) \leq E^*. \tag{39}$$

Inasmuch as $h(x)$ is discontinuous, the usual Tangency Condition does not hold, so that the inequalities (38), (39) must be enforced directly.

It is to be noted that putting an upper bound on h prevents the breakdown of the solution when $\psi(x) = 0$, leading to $H(k) = 0$ in (11) and $h(x) = \infty$ in (22).

THE SUFFICIENCY CONDITION

For a strong relative minimum the Sufficiency Conditions[8)] are I, II'_n, III', IV, described below. The Multiplier Rule, I, is satisfied by the solution of the Euler equation provided λ_1 and λ_2 are not both zero and the Transversality Condition is satisfied. The latter, in our problem reduces to

$$\left| g_t + F_{t'} \right|^{x_2} = 0 , \tag{40}$$

or

$$\lambda_2(x_2) = 1. \tag{41}$$

The Weierstrauss Condition, II_n', expressed in terms of the E-function, is

$$E(z',Z') = F(x,z,Z') - F\,(x,z,z') - (Z'-z')F_{z'}(x,z,z') > 0, \tag{42}$$

where the slope functions z' are functions of the field coordinates x,z in the neighborhood of the solution and $Z' \neq z'$. If the derivatives of certain variables z_k are lacking in F, then such z_k must be classed a slope function and (42) modified accordingly. In our problem, therefore, the slope functions are h, S', t', and (42) becomes, in view of (30), (22)

$$E = \lambda_1 \psi(0) H^k\ \delta h^{k-1} + \lambda_2 \delta h^{-1}$$

$$= \frac{\lambda_2}{(k-1)h(1+\beta)} \left[(1 + \beta)^k - 1 - k\,\beta \right] > 0, \tag{43}$$

where δh denotes a strong variation and

$$\beta \equiv \delta h/h. \tag{44}$$

We observe that $h > 0$, $\beta > -1$, $k \geq 2$, and

$$(1 + \beta)^k - 1 - k\beta > 0 \quad \text{if } - 1 < \beta \neq 0,\ k > 0 \tag{45}$$

Thus (42) is equivalent to the requirement

$$\lambda_2(x) > 0, \tag{46}$$

which is obviously satisfied, since

$$\lambda_2(x) \equiv 1 \tag{47}$$

in virtue of (41) and (21).

The Clebsh Condition, III', is

$$F_{z'z'}\delta z'_i \delta z'_j > 0 \tag{48}$$

for all $\delta z'_i$ satisfying the differentiated equations of constraint. In

our problem $h(x)$ is the only slope function contributing to the quadratic form, so that (48) becomes

$$\lambda_2 k h^{-3}\,\delta h^2 > 0, \tag{49}$$

which is automatically satisfied, since

$$\lambda_2 = 1, \quad k \geq 2, \; h > 0.$$

The positive-definiteness of the second variation, which is required by Condition IV', in our problem reduces to

$$d^2 J = \int_0^X \lambda_2 k h^{-3}\,\delta h^2 dx > 0, \tag{50}$$

since x_1 and x_2 are fixed, while g and the end-functions are linear in their arguments. Since all the conditions are satisfied, the unique solution (22) furnishes an absolute minimum of t.

AN APPROXIMATE SOLUTION

If f_y, $y^{(k)}$, $c(x)$ are replaced by some constant bounds defined by means of

$$\left| f_y \right| \leq L\,, \quad \left| y^k(x) \right| \leq ML^{k-1}, \quad \left| c(x) \right| \leq c, \tag{51}$$

then (10) becomes

$$\psi(x) = cML^{k-1} = \psi(0) \tag{52}$$

In terms of the dimensionless quantities

$$x' = Lx, \quad h' = Lh, \tag{53}$$

(11), (12) reduce to

$$H = e^{-x/k}, \; S = E(x)e^{-x} \tag{54}$$

on dropping the primes, while (22-25) become, respectively

$$h = h(0)e^{x/k}$$

$$h(0) = \left| \frac{LE*}{Mc} \; \frac{e^{-X}}{k(1-e^{-X/k})} \right|^{\frac{1}{k-1}} ,$$

$$t(\bar{X}) = \frac{k}{\bar{h}(0)} \quad (1 - e^{-\bar{X}/k}), \tag{55}$$

$$\bar{h} = \left| \frac{LE^*}{Mc} \quad \frac{e^{-\bar{X}}}{1 - e^{-\bar{X}}} \right|^{\frac{1}{k-1}},$$

$$t(X) = \bar{X}/\bar{h} \,,$$

for the discontinuous solution (28), (36), (37) become, respectively,

$$h_*(x) = h_*(0)2^{-m(x)} \;;$$

$$m(x) = \left[x/k \log 2 \right],$$

$$h_*(0) = \left| \frac{LE^* e^{-X}}{Mc \; 2(1-2^{-k}) \; (1-e^{-X/k})} \right|^{\frac{1}{k-1}}, \tag{56}$$

$$t_* = \frac{k \log 4}{h_*(0)} \quad (1 - e^{-X/k}).$$

A comparison of the discontinuous and the continuous solutions yields

$$h_* (x) : h(x) = 2^m e^{-x/k} f(k), \tag{57}$$

$$t_* : t = f^{-1} (k) \log 4,$$

where f(k), defined by

$$f(k) \equiv \left| k/2(1-2^{-k}) \right|^{\frac{1}{k-1}} \tag{58}$$

is a weak function of k. It has the properties

$$f(0) = \log 4, \; f(1) = e/2, \; f(\infty) = 1, \quad f' < 0 \tag{59}$$

and is tabulated below.

Table 2

k	0	1	2	3	4	5	6
f(k)	1.39	1.36	1.33	1.31	1.29	1.27	1.25
$t^*:t$	1.00	1.02	1.05	1.06	1.08	1.09	1.11

In the range $2 \leq k < \infty$

$$1 \leq f(k) < 4/3, \tag{60}$$

while $2^m e^{-x/k}$ is a periodic piecewise monotonic function oscillating between the values 1/2 and 1 with the period $x = k \log 2$. Consequently, $h_*: h$ oscillates in the range

$$1/2 \leq f(k)/2 \leq h_*: \ h \leq f(k) < 4/3, \tag{61}$$

so that the continuous and the discontinuous solutions are interlaced.

If X is approximated by an integral multiple of the period k log 2, we may put

$$X = N\, k \log 2 \tag{62}$$

and deduce from (55), (56)

$$\begin{aligned} t_*: \tau &= \frac{2(1-2^{-N})}{N} \left| \frac{2(1-2^{-k})\,(1-2^{-N})}{1-2^{-Nk}} \right|^{\frac{1}{k-1}} \\ &\equiv \phi(N,k). \end{aligned} \tag{63}$$

The following special values and an asymptotic expansion of $\phi(N,k)$ are to be noted:

$$\phi(1,k) = 1, \quad \phi(\infty,k) = 0,$$

$$\phi(N,k) = \frac{2}{N} \left| 2(1-2^{-k}) \right|^{\frac{1}{k-1}} \quad \text{if} \quad N \to \infty \tag{64}$$

Consequently, as N ranges from 1 to ∞ the "relative gain", $1 - \phi$, of the optimum mesh, $h_*(x)$, in comparison with the constant mesh, $\bar{h}$, ranges from 0 to 1. It should be noted, however, that the use of constant bounds in (51) may lead to a solution h(x) which is fantastically smaller than the one based on the local bound $\epsilon(x)$, varying from point to point. For this reason the results of this section must be used with caution.

ON THE CHOICE OF METHOD OF INTEGRATION

If we do not restrict ourselves to a particular method, the preceding results enable us to choose the optimum k. By making the reasonable assumption that the time of integration per step is proportional to the order of accuracy, k-1, we can write t as an explicit function of k, in the form

$$t = (k-1) \int_0^X H_k(\xi)d\xi \left| \int_0^X H_k(\xi)d\xi\, \psi_k(0)/S^* \right|^{\frac{1}{k-1}} \tag{65}$$

where H_k is a known function of x, and proceed to minimize t with respect

to k. For example, if $H_k(x)$, $\psi_k(0)$ should be constants independent of k, the optimum $k = k_*$ is

$$k_* = \left[1 + \log A\right] ,$$

$$A \equiv \psi_k(0) H_k X / S^* \tag{66}$$

As one would naturally expect, k_* increases with the range X and with the precision required.

In iterative procedures the number, $\nu(x)$, of cycles per step can also be optimized. This can be achieved by introducing ν as a factor of t and regarding $\epsilon(h,\nu)$ as a known function of h, ν, as well as x. Then h(x) and $\nu(x)$ are determined from the Euler equations.

$$h\epsilon_h + \nu\epsilon_\nu = 0 \quad ,$$

$$\log\,(h^2\,\epsilon_h/\nu) = \int_0^X f_y(\xi)\,d\xi + \text{const.} \tag{67}$$

A SYSTEM OF DIFFERENTIAL EQUATIONS

The construction of optimum mesh can be extended to a system

$$y_i(x) = f_i(x, y_j) \;,\quad y_i(0) = y_{io}, \tag{68}$$

$$i, j = 1, \ldots n.$$

The Lagrangian can now be written

$$F = \bar{\mu}(\epsilon + JE - E') + \lambda\,(h^{-1} - t'), \tag{69}$$

where J is the Jacobian matrix

$$J_{ij} = \partial f_i / \partial y_j, \tag{70}$$

E and ϵ are column matrices satisfying the relations

$$\delta y_i \leq E_i \;,$$

$$\epsilon_i = h^{k-1}\,\psi_i \tag{71}$$

and $\tilde{\mu}(x)$ is a row of Lagrange multipliers. Generally, in this section a bar placed above a letter will denote matrix transposition. Let the end-conditions be

$$E(0) = 0,\ t(0) = 0,\ \Phi = \left|E(X)\right| - E^* = 0 \tag{72}$$

The Euler equations yield

$$\bar{\mu}' = -\bar{\mu}\ J,$$

$$h = h\ (0)\ H^{-1} \tag{73}$$

$$H \equiv \left|\bar{\mu}x/\bar{\mu}(0)\ \psi(0)\right|^{1/k}.$$

The solution of (73.1) is

$$\bar{\mu}(x) = \bar{\mu}(0)\ Z\ (x), \tag{74}$$

where Z is the matrix solution of

$$Z' = -Z\ J,\ Z(0) = I, \tag{75}$$

I being the unit matrix. Since J and Z become known functions of x as soon as y(x) is known, it remains to determine the n independent constants $\mu_i(0)$, h(0), satisfying the constraint

$$\left|\mu(0)\right| \equiv \left|\sum \mu^2_i\ (0)\right|^{1/2} = 1. \tag{76}$$

The latter condition may be imposed without any loss of generality, since (74) is a linear homogeneous equation. $\mu(0)$ can be found with the aid of the Transversality condition

$$\Gamma_{E_i} + F_{E_i'} \Big|^{X} = 0, \tag{77}$$

where

$$\Gamma = g + \alpha\,\Phi \quad = t + \alpha\left(\left|E\right| - E^*\right), \tag{78}$$

α being a constant Lagrange multiplier. From (77),(78), (69) follows

$$\mu(x) = \alpha\ E(x)/\left|E(x)\right|, \tag{79}$$

which, in view (75), (77) yields

$$\bar{\mu}(0) = E(X)Z^{-1}(X)\Big/\left|E(X)Z^{-1}(X)\right|. \tag{80}$$

Although E(X) appearing in (80) is not immediately available, it is possible to calculate an approximation by extrapolating y(X) to "zero-mesh". An iterative process may then be set up between (80) and (73), which may converge to yield the value of $\mu(0)$. Finally, h(0) may be determined from the end-conditions, with the aid of

$$E(x) = Z^{-1}(x) \int_0^x Z(\xi)\,\psi(\xi)\ h^{k-1}(\xi)\ d\xi \tag{81}$$

and of (73), as

$$h(0) = \left\{ Z(X)E(X) \Big/ \int_0^X \frac{Z(\xi)\ \psi\ (\xi)\ \ d\xi}{|\bar{\mu}(0)Z(\xi)\ \psi\ (\xi)\,|^{1-1/k}} \right\}^{\frac{1}{k-1}} . \tag{82}$$

Obviously, if $n = 1$

$$J = f_y,\ Z = \exp\left(-\int_0^X f_y\, d\xi\right),\ \ \mu(0) = 1 \tag{83}$$

so that (81), (82), and (73.3) reduce to (5) (23) and (11), respectively.

NUMERICAL EXAMPLE

Required the numerical solution of $y' = 2y - e^x$, $y(0) = 1$ by the method of Runge-Kutta, with a precision of $E^* = 1$ at $X = 10.4$. Since the formal solution is known to be $y = e^x$, it is possible to write down the theoretical optimum mesh. Here $\psi(x) = c(x)e^x$, $\psi(0) = c(0)$ From Table 1, $f_y = 2$ so that from (11), (12), (36), (37) there follows:

$$H = e^{-x/k}\ ,\ S^* = e^{-2x},\ m = \left[x/k \log 2\right],$$

$$h_*(0) = \left| \frac{S^*}{c(0)\ 2\ (1-2^{-k})\ (1-2^{-N})} \right|^{\frac{1}{k-1}} \tag{84}$$

$$t_* = \frac{1}{h_*(0)} \quad (2_k \log 2)\ (1 - 2^{-N}\ ),$$

and from (25)

$$\bar{h} = \left| \frac{S^*}{c(0)\ (1-2^{-Nk})} \right|^{\frac{1}{k-1}} \tag{85}$$

$$\bar{t} = X/\bar{h}\ .$$

In our example $k = 5$; hence $N = 3$ from (62); $c(x) = 0.073 = c(0)$ from formula (13) of reference 3). The numerical values now become:

$$m(x) = \left[x/3.47 \right], \tag{86}$$

$$h_*(0) = 0.0093\ ,$$

$$t_* = 651 \text{ steps},$$

$$\bar{h} = 0.0106\ ,$$

$$\bar{t} = 981 \text{ steps},$$

giving the optimum mesh a relative gain of 34%.

Of course, in numerical work the solution $y(x)$ is not known a priori. The optimum mesh can be constructed by a computing machine, using a rough integration of the differential equation $y' = f(x,y)$ with an arbitrary initial $h(0)$, combined with the following algorithm:

$$\Delta m = \left[\left(\int_0^x f_y(\xi)d\xi - \log \eta/\eta(0) \right) \Big/ k \log 2 \right],$$

$$\eta = h^k \psi = (\Delta y^{(1)} - \Delta y^{(2)}/(1-2)^{-k+1}),$$

$$m(x_{i+1}) = m(x_i) + \Delta m,$$

$$h = h(0)\, 2^m, \tag{87}$$

$$x_{i+1} = x_i + h(x_i).$$

Here η is an estimate of the local truncation error, which is calculated by means of extrapolation to "zero mesh"[9]. $\Delta y^{(1)}$ and $\Delta y^{(2)}$ are the increments of y due to step h and a succession of two half-steps, $h/2$. (87.1) is a consequence of (36) and (11). The constant $h(0)$ satisfying the condition $E(x) = E^*$ can be found by some suitable Error Control procedure[10].

CONCLUSIONS

We have shown that an optimum mesh exists and have described the process whereby it could be constructed. Generally, the saving of time depends on the differential equation to be integrated, the method used, and the range of integration. As would be expected, the saving increases with the range. If the latter is sufficiently great, the gain will exceed the cost of additional labor necessary for the construction of the optimum mesh, and will fully justify its use.

BORIS GARFINKEL

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